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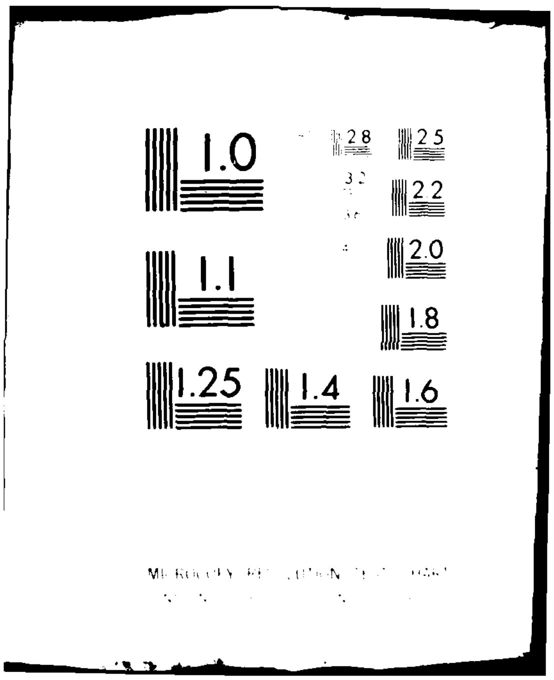
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RANK-ORDER TESTS FOR THE PARALLELISM OF SEVERAL REGRESSION SURFACES*

AFOSR-76-2927

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For testing the hypothesis that several (s ≥ 2) linear regression surfaces $\underset{\sim}{X}_{ki} = \underset{\sim}{\alpha}_k + \underset{\sim}{\beta}_k \underset{\sim}{c}_{ki} + \underset{\sim}{Z}_{ki}$ (k = 1,...,s) are parallel to one another, i.e., $\underset{\sim}{\beta}_1 = \dots = \underset{\sim}{\beta}_s$, a class of rank-order tests are considered. The tests are shown to be asymptotically distribution-free, and their asymptotic efficiency relative to the general likelihood ratio test is derived. Asymptotic optimality in the sense of Wald is also discussed.

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0. <u>Introduction</u>. Consider $s\ (\geq 2)$ linear regression models

$$(0.1) \qquad X_{ki} = \alpha_k + \underset{\sim}{\beta}_k' \underset{\sim}{c}_{ki} + Z_{ki} , \quad i = 1,\ldots,n_k \ ; \ k = 1,\ldots,s$$

where, for each $k = 1,\ldots,s$, α_k is the (unknown) intercept,

$$(0.2) \qquad \underset{\sim}{\beta}_k = (\beta_{k1},\ldots,\beta_{kq})'$$

is a q-dimensional vector of unknown regression parameters,

$$(0.3) \qquad \underset{\sim}{c}_{ki} = (c_{k1i},\ldots,c_{kqi})'$$

is a q-dimensional vector of known regression constants for each $i = 1,\ldots,n_k$, and the Z_{ki} are all independent (error) random variables with the same (but unknown) continuous distribution

$$(0.4) \qquad F(x) = P(Z_{ki} \leq x) \ , \quad k = 1,\ldots,s \ ; \ i = 1,\ldots,n_k \ .$$

A problem of interest is that of testing whether the s regression surfaces are parallel to one another, i.e.,

$$(0.5) \qquad H_o : \underset{\sim}{\beta}_1 = \ldots = \underset{\sim}{\beta}_s = \underset{\sim}{\beta}\text{(unknown)}$$

vs.

$$H : \underset{\sim}{\beta}_k \neq \underset{\sim}{\beta}_j \quad \text{for some} \quad 1 \leq k \neq j \leq s \ .$$

For the special case $q = 1$, i.e., testing the parallelism of several regression lines, Sen (1969) has proposed a class of rank order tests. In the present paper we study the problem in the general case $q \geq 1$. Preliminary notations and assumptions are given in section 1. In section 2 a class of asymptotically distribution-free aligned rank-order tests are proposed. These are univariate counterparts of a multivariate problem briefly mentioned in Sen and Puri (1977) but not solved. The asymptotic distribution of the test statistics is derived in section 3. In section 4 we derive the asymptotic relative efficiency of the proposed tests with respect to the general likelihood ratio test of the same problem. Finally, in section 5, asymptotic optimality in the sense of Wald (1943) is discussed.

1. Preliminary Notations and Assumptions. For each $k = 1,\dots,s$ let

$$(1.1) \qquad \bar{c}_{kn_k} = n_k^{-1} \sum_{i=1}^{n_k} c_{ki} = (\bar{c}_{k1n_k},\dots,\bar{c}_{kqn_k})'$$

where

$$(1.2) \qquad \bar{c}_{kmn_k} = n_k^{-1} \sum_{i=1}^{n_k} c_{kmi} \ , \quad m = 1,\dots,q \ .$$

We assume that the $q \times q$ symmetric matrices

$$(1.3) \qquad M_{kn_k} = \sum_{i=1}^{n_k} (c_{ki} - \bar{c}_{kn_k})(c_{ki} - \bar{c}_{kn_k})' \ , \ k = 1,\dots,s$$

are positive definite and that the limiting matrices

$$(1.4) \qquad M_k = \lim_{n_k \to \infty} n_k^{-1} M_{kn_k} \ , \ k = 1,\dots,s$$

exist and are positive definite. Simplifying some of Jurečková's

(1971) conditions on the regression constants, we also assume that each $\underset{\sim}{c}_{ki}$ can be expressed as a difference

$$(1.5)\qquad \begin{aligned} \underset{\sim}{c}_{ki} &= \underset{\sim}{c}_{ki(1)} - \underset{\sim}{c}_{ki(2)}\ , \\ \underset{\sim}{c}_{ki(j)} &= (c_{k1i(j)},\ldots,c_{kqi(j)})' \end{aligned}$$

where, for each $k = 1,\ldots,s$, $m = 1,\ldots,q$ and $j = 1,2$, $c_{kmi(j)}$ is nondecreasing in i , and the $c_{kmi(j)}$'s satisfy

$$(1.6)\qquad \begin{aligned} &\lim_{n_k\to\infty} n_k^{-1} \max_{1\le i\le n_k} [c_{kmi(j)} - \bar{c}_{kmn_k(j)}]^2 = 0\ , \\ &\bar{c}_{kmn_k(j)} = n_k^{-1} \sum_{i=1}^{n_k} c_{kmi(j)} \end{aligned}$$

and

$$(1.7)\qquad \lim_{n_k\to\infty} n_k^{-1} \sum_{i=1}^{n_k} [c_{kmi(j)} - \bar{c}_{kmn_k(j)}]^2 \in (0,\infty)\ ,$$

which together imply the Noether condition

$$(1.8)\qquad \lim_{n_k\to\infty} \{ \max_{1\le i\le n_k} [c_{kmi(j)} - \bar{c}_{kmn_k(j)}]^2 \Big/ \sum_{i=1}^{n_k} [c_{kmi(j)} - \bar{c}_{kmn_k(j)}]^2 \} = 0\ .$$

We denote the total (combined) sample size by

$$(1.9)\qquad N = \sum_{k=1}^{s} n_k$$

and assume that the limits

(1.10) $$r_k = \lim_{N\to\infty} (n_k/N) \ , \quad k = 1,\ldots,s$$

exist and satisfy

(1.11) $$r_o \le r_k \le 1 - r_o \ , \ k = 1,\ldots,s$$

for some $0 < r_o < 1/s$. Thus we have

(1.12) $$\lim_{N\to\infty} n_k = \infty \ , \ k = 1,\ldots,s$$

and

(1.13) $$\sum_{k=1}^{s} r_k = 1 \ ,$$

and the matrices

(1.14) $$M_k^* = \lim_{N\to\infty} N^{-1} M_{kn_k} = \lim_{N\to\infty} (n_k/N) M_k = r_k M_k \ , \ k = 1,\ldots,s$$

are symmetric and positive definite.

For each positive integer n , let the scores $a_n(1),\ldots,a_n(n)$ be generated by a non-constant and square integrable function ψ on $(0,1)$ according to one of the following two ways:

(1.15) $$a_n(i) = \psi[i/(n+1)] \ , \quad i = 1,\ldots,n$$

or

(1.16) $$a_n(i) = E[\psi(U_{ni})] \ , \quad i = 1,\ldots,n$$

where $U_{n1} \le \ldots \le U_{nn}$ are the order statistics of a random sample of size n from the uniform distribution over $(0,1)$. We assume that ψ can be expressed as the difference $\psi = \psi_1 - \psi_2$ of two non-decreasing and absolutely continuous functions ψ_1 and ψ_2 on $(0,1)$. Let

$$(1.17) \qquad \lambda(\psi) = \{\int_0^1 [\psi(u) - \bar{\psi}]^2 \, du\}^{1/2} , \quad \bar{\psi} = \int_0^1 \psi(u)du .$$

Thus we have $0 < \lambda(\psi) < \infty$.

We assume that the underlying distribution function F has an absolutely continuous density $f = F'$ with finite positive Fisher information

$$(1.18) \qquad 0 < I(f) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 dF(x) < \infty .$$

We note that

$$(1.19) \qquad I(f) = [\lambda(\phi_f)]^2 = \int_0^1 [\phi_f(u)]^2 du$$

where

$$(1.20) \qquad \phi_f(u) = -f'[F^{-1}(u)]/f[F^{-1}(u)] , \quad u \in (0,1)$$

with

$$(1.21) \qquad \bar{\phi}_f = \int_0^1 \phi_f(u)du = 0 .$$

2. <u>The Proposed Rank-order Tests</u>. For $\underset{\sim}{b} = (b_1,\ldots,b_q) \in \mathbb{R}^q$ and $k = 1,\ldots,s$, let

(2.1) $R_{kin_k}(\underset{\sim}{b})$ = the rank of $X_{ki} - \underset{\sim}{b}\underset{\sim}{c}_{ki}$ among $X_{k1} - \underset{\sim}{b}\underset{\sim}{c}_{k1},\ldots,X_{kn_k} - \underset{\sim}{b}\underset{\sim}{c}_{kn_k}$ in the ascending order of magnitude, (2.2) $$S_{kmn_k}(\underset{\sim}{b}) = \sum_{i=1}^{n_k} (c_{kmi} - \bar{c}_{kmn_k})\, a_{n_k}[R_{kin_k}(\underset{\sim}{b})],\ m = 1,\ldots,q$$

where $a_{n_k}(1),\ldots,a_{n_k}(n_k)$ are generated according to (1.15) or (1.16) (with n replaced by n_k),

(2.3) $$\underset{\sim}{S}_{kn_k}(\underset{\sim}{b}) = (S_{k1n_k}(\underset{\sim}{b}),\ldots,S_{kqn_k}(\underset{\sim}{b}))\ ,$$

and define

(2.4) $$\underset{\sim}{S}_N(\underset{\sim}{b}) = \sum_{k=1}^{s} \underset{\sim}{S}_{kn_k}(\underset{\sim}{b}) = (S_{N1}(\underset{\sim}{b}),\ldots,S_{Nq}(\underset{\sim}{b}))\ .$$

Let

(2.5) $$B_{(N)} = \{\underset{\sim}{b} \in \mathbb{R}^q : \sum_{m=1}^{q} |S_{Nm}(\underset{\sim}{b})| = \text{minimum}\}$$

and choose one element

(2.6) $$\hat{\underset{\sim}{\beta}}_N \in B_{(N)}$$

as an estimate of $\underset{\sim}{\beta}$. Define the s vectors of aligned rank statistics

(2.7) $$\hat{\underset{\sim}{S}}_{Nk} = \underset{\sim}{S}_{kn_k}(\hat{\underset{\sim}{\beta}}_N),\quad k = 1,\ldots,s$$

and let

(2.8) $$\lambda_N = \{N^{-1} \sum_{k=1}^{s} \sum_{i=1}^{n_k} [a_{n_k}(i) - \bar{a}_{n_k}]^2\}^{\frac{1}{2}}$$

where

(2.9) $$\bar{a}_{n_k} = n_k^{-1} \sum_{i=1}^{n_k} a_{n_k}(i)\ ,\quad k = 1,\ldots,s\ .$$

Then a class of aligned rank-order tests of (0.5), each determined by a score-generating function ψ , can be based on the statistics

$$(2.10) \qquad Q_N = \lambda_N^{-2} \sum_{k=1}^{s} \hat{\underset{\sim}{S}}_{Nk} M_{kn_k}^{-1} \hat{\underset{\sim}{S}}'_{Nk} ,$$

whose asymptotic distribution under H_o is given by Theorem 2.1, which in turn follows from Theorem 3.1 (see section 3).

<u>Theorem 2.1</u>. <u>Under</u> H_o , Q_N <u>has asymptotically the (central) chi-square distribution</u> $\chi^2_{(s-1)q}$ <u>with</u> $(s - 1)q$ <u>degrees of freedom</u>.

For $0 < \varepsilon < 1$, let $\chi^2_{(s-1)q,\varepsilon}$ be the upper $100\varepsilon\%$ point of the $\chi^2_{(s-1)q}$ distribution. Then for large N we have the following asymptotically distribution-free test of approximately size ε :

(2.11) Reject H_o (in favor of H) if and only if

$$Q_N \geq \chi^2_{(s-1)q,\varepsilon} .$$

3. <u>Asymptotic Distribution of the Test Statistics</u>. Consider the sequence of hypotheses

$$(3.1) \qquad H_N : \underset{\sim}{\beta}_k = \underset{\sim}{\beta}_{kN} = \underset{\sim}{\beta} + N^{-\frac{1}{2}} \underset{\sim}{b}^*_k , \quad k = 1,\ldots,s$$

where the s vectors $\underset{\sim}{b}^*_k \in \mathbb{R}^q$, $k = 1,\ldots,s$ are such that

$$(3.2) \qquad \sum_{k=1}^{s} \underset{\sim}{b}^*_k M^*_k = \underset{\sim}{0} .$$

Then the asymptotic distribution of Q_N under H_N is given by the following theorem.

Theorem 3.1. *Under* H_N, Q_N *has asymptotically the non-central chi-square distribution* $\chi^2_{(s-1)q}(\Delta_Q)$ *with* $(s-1)q$ *degrees of freedom and noncentrality parameter*

$$(3.3) \qquad \Delta_Q = [\gamma(\psi,f)/\lambda(\psi)]^2 \sum_{k=1}^{s} \underset{\sim}{b}_k^* M_k^* \underset{\sim}{b}_k^{*\prime}$$

where

$$(3.4) \qquad \gamma(\psi,f) = \int_0^1 \psi(u)\phi_f(u)\,du \ .$$

Remark. Clearly for $\underset{\sim}{b}_1^* = \dots = \underset{\sim}{b}_s^* = \underset{\sim}{0}$, which satisfy (3.2), H_N reduces to H_O, and Δ_Q reduces to 0. Thus Theorem 2.1 is a special case of Theorem 3.1.

For later purpose we also estimate the $\underset{\sim}{\beta}_k$'s separately. For each $k = 1,\dots,s$, let

$$(3.5) \qquad B_{kn_k} = \{\underset{\sim}{b} \in \mathbb{R}^q : \sum_{m=1}^{q} | S_{kmn_k}(\underset{\sim}{b}) | = \text{minimum}\}$$

and choose one element

$$(3.6) \qquad \hat{\underset{\sim}{\beta}}_{kn_k} \in B_{kn_k}$$

as an estimate of $\underset{\sim}{\beta}_k$ based on the k-th sample

$$(3.7) \qquad \underset{\sim}{X}_{kn_k} = (X_{k1},\dots,X_{kn_k}) \ .$$

We note that since the s samples $\underset{\sim}{X}_{1n_1},\dots,\underset{\sim}{X}_{sn_s}$ are independent so are the estimates $\hat{\underset{\sim}{\beta}}_{1n_1},\dots,\hat{\underset{\sim}{\beta}}_{sn_s}$. By Jurečková's (1971) results (see Theorems 3.1 and 4.1, and Lemmas 4.1 and 4.5), the distribution (denoted by $\mathcal{D}$) of $n_k^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta}_k)$ is asymptotically normal (denoted by $\mathcal{N}_q$), i.e.,

$$(3.8) \quad \mathcal{D}[n_k^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta}_k)] \to \mathcal{N}_q(\underset{\sim}{0}, [\lambda(\psi)/\gamma(\psi,f)]^2 M_k^{-1}) \quad (k = 1,\dots,s) ,$$

and

$$(3.9) \quad n_k^{-\frac{1}{2}} \underset{\sim}{S}_{kn_k}(\hat{\underset{\sim}{\beta}}_{kn_k}) = o_p(1) \quad (k = 1,\dots,s) .$$

Similarly we have

$$(3.10) \quad N^{-\frac{1}{2}} \underset{\sim}{S}_N(\hat{\underset{\sim}{\beta}}_N) = o_p(1) .$$

We need the following lemmas to prove Theorem 3.1.

Lemma 3.2. For each $k = 1,\dots,s$ we have

$$(3.11) \quad N^{-\frac{1}{2}} \hat{\underset{\sim}{S}}_{Nk} = \gamma(\psi,f) N^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \hat{\underset{\sim}{\beta}}_N) M_k^* + o_p(1) .$$

Proof. By Theorem 3.1 of Jurečková (1971), for each $k = 1,\dots,s$ we have

$$(3.12) \quad \begin{aligned} n_k^{-\frac{1}{2}} \underset{\sim}{S}_{kn_k}(\hat{\underset{\sim}{\beta}}_{kn_k}) &= n_k^{-\frac{1}{2}} \underset{\sim}{S}_{kn_k}(\underset{\sim}{\beta}_k) \\ &\quad - \gamma(\psi,f) n_k^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta}_k) M_k + o_p(1) \end{aligned}$$

and

$$(3.13) \quad \begin{aligned} n_k^{-\frac{1}{2}} \underset{\sim}{S}_{kn_k}(\hat{\underset{\sim}{\beta}}_N) &= n_k^{-\frac{1}{2}} \underset{\sim}{S}_{kn_k}(\underset{\sim}{\beta}_k) \\ &\quad - \gamma(\psi,f) n_k^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\beta}_k) M_k + o_p(1) . \end{aligned}$$

Subtracting (3.12) from (3.13) and using (2.7) and (3.9), we have

$$(3.14)\qquad n_k^{-\frac{1}{2}}\hat{\underset{\sim}{S}}_{Nk} = \gamma(\psi,f)\, n_k^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \hat{\underset{\sim}{\beta}}_N)M_k + o_p(1) \ .$$

Multiplying both sides of (3.14) by $(n_k/N)^{\frac{1}{2}}$ and using (1.14), we obtain (3.11).

For later use we also define the $q \times q$ matrix

$$(3.15)\qquad D = \sum_{k=1}^{s} M_k^* = (d_{m\ell})_{m,\ell=1,\ldots,q}$$

which, being a sum of symmetric and positive definite matrices, is itself symmetric and positive definite and hence has a symmetric inverse

$$(3.16)\qquad A = D^{-1} = (a_{\ell m})_{\ell,m=1,\ldots,q} \ .$$

Thus we have

$$(3.17)\qquad DA = AD = \sum_{k=1}^{s} M_k^* A = A\sum_{k=1}^{s} M_k^* = I_q \ ,$$

where I_q is the $q \times q$ identity matrix.

<u>Notation</u>. Let $\{\underset{\sim}{U}_n\}$ and $\{\underset{\sim}{V}_n\}$ be two sequences of random vectors of the same dimension. Then

$$(3.18)\qquad \underset{\sim}{U}_n \sim \underset{\sim}{V}_n \ \text{ if and only if } \ \underset{\sim}{U}_n - \underset{\sim}{V}_n = o_p(1) \ .$$

<u>Lemma 3.3</u>.

$$(3.19)\qquad N^{\frac{1}{2}}\hat{\underset{\sim}{\beta}}_N \sim N^{\frac{1}{2}} \sum_{k=1}^{s} \hat{\underset{\sim}{\beta}}_{kn_k} M_k^* A \ .$$

Proof. By (2.4), (2.7), (3.11) and (3.15) we have

$$N^{-\frac{1}{2}}S_N(\hat{\beta}_N) = \sum_{k=1}^{s} N^{-\frac{1}{2}}\hat{S}_{Nk}$$

$$= \sum_{k=1}^{s} \gamma(\psi,f)N^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \hat{\beta}_N)M_k^* + o_p(1)$$

$$= \gamma(\psi,f)[N^{\frac{1}{2}}\sum_{k=1}^{s} \hat{\beta}_{kn_k}M_k^* - N^{\frac{1}{2}}\hat{\beta}_N D] + o_p(1) .$$

Since $\gamma(\psi,f)$ is a non-zero constant, by (3.10) we have

$$N^{\frac{1}{2}}\hat{\beta}_N D \sim N^{\frac{1}{2}}\sum_{k=1}^{s}\hat{\beta}_{kn_k}M_k^* \quad , \tag{3.20}$$

which, together with (3.17), implies (3.19).

Lemma 3.4. Under H_N, for each $k = 1,\ldots,s$ we have

$$\mathcal{D}[N^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta)|H_N] \to N_q(b_k^*,[\lambda(\psi)/\gamma(\psi,f)]^2 M_k^{*-1}) . \tag{3.21}$$

Proof. By (3.1), under H_N for each $k = 1,\ldots,s$ we have

$$n_k^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta) = n_k^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta_k) + (n_k/N)^{\frac{1}{2}}b_k^*$$

and so by (1.10) and (3.8) we have

$$\mathcal{D}[n_k^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta)|H_N] \to N_q(r_k^{\frac{1}{2}}b_k^*,[\lambda(\psi)/\gamma(\psi,f)]^2 M_k^{-1}) .$$

Hence, by (1.14), under H_N the random vector

$$N^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta) = (n_k/N)^{-\frac{1}{2}} n_k^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \beta)$$

is asymptotically q-variate normal with mean b_k^* and covariance matrix

$$[\lambda(\psi)/\gamma(\psi,f)]^2 r_k^{-1} M_k^{-1} = [\lambda(\psi)/\gamma(\psi,f)]^2 M_k^{*-1} .$$

<u>Lemma 3.5</u>. <u>Under</u> H_N, <u>the</u> sq-<u>dimensional random vector</u>

(3.22) $$T_N = N^{\frac{1}{2}}(\hat{\beta}_{1n_1} - \hat{\beta}_N, \ldots, \hat{\beta}_{sn_s} - \hat{\beta}_N)$$

<u>is asymptotically normal</u> $N_{sq}(b^*, [\lambda(\psi)/\gamma(\psi,f)]^2 J)$, <u>where</u>

(3.23) $$b^* = (b_1^*, \ldots, b_s^*)$$

<u>and</u> J <u>can be partitioned as</u>

(3.24) $$J = (J_{kj})_{k,j=1,\ldots,s}$$

<u>with</u>

(3.25) $$J_{kj} = \delta_{kj} M_j^{*-1} - A$$

(δ_{kj} <u>being the Kronecker delta</u>).

<u>Proof</u>. We prove Lemma 3.5 by showing that any linear combination of the components of T_N is asymptotically normal under H_N, with the appropriate mean and variance. Let

(3.26) $$t = (t_1, \ldots, t_s) \in \mathbb{R}^{sq}$$

where

(3.27) $$\underset{\sim}{t}_k = (t_{k1}, \ldots, t_{kq}) \in \mathbb{R}^q \quad , \quad k = 1, \ldots, s \ ,$$

and let

(3.28) $$\underset{\sim}{u} = \sum_{k=1}^{s} \underset{\sim}{t}_k \ .$$

Then by (3.19) we have

$$\underset{\sim}{T}_N \underset{\sim}{t}' = \sum_{k=1}^{s} N^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \hat{\underset{\sim}{\beta}}_N)\underset{\sim}{t}_k'$$

(3.29) $$= \sum_{k=1}^{s} N^{\frac{1}{2}}\hat{\underset{\sim}{\beta}}_{kn_k}\underset{\sim}{t}_k' - N^{\frac{1}{2}}\hat{\underset{\sim}{\beta}}_N \underset{\sim}{u}'$$

$$\sim \sum_{k=1}^{s} N^{\frac{1}{2}}\hat{\underset{\sim}{\beta}}_{kn_k}\underset{\sim}{t}_k' - N^{\frac{1}{2}} \sum_{k=1}^{s} \hat{\underset{\sim}{\beta}}_{kn_k} M_k^* A \underset{\sim}{u}' \ .$$

By making the substitution $\hat{\underset{\sim}{\beta}}_{kn_k} = (\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta}) + \underset{\sim}{\beta}$ on the right-hand side of $\sim$ in (3.29) and then making cancellation (using (3.17)), we have

(3.30) $$\underset{\sim}{T}_N \underset{\sim}{t}' \sim \sum_{k=1}^{s} N^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta})(\underset{\sim}{t}_k' - M_k^* A \underset{\sim}{u}') \ .$$

Now by (3.21) and the symmetry of M_k^* and A, under H_N the random variable

$$N^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{kn_k} - \underset{\sim}{\beta})(\underset{\sim}{t}_k' - M_k^* A \underset{\sim}{u}')$$

is asymptotically normal with mean $\underset{\sim}{b}_k^*(\underset{\sim}{t}_k' - M_k^* A \underset{\sim}{u}')$ and variance

$$\{\lambda(\psi)/\gamma(\psi, f)\}^2 (\underset{\sim}{t}_k - \underset{\sim}{u} A M_k^*) M_k^{*-1} (\underset{\sim}{t}_k' - M_k^* A \underset{\sim}{u}') \ .$$

So, by independence of the $\hat{\underset{\sim}{\beta}}_{kn_k}$'s, under H_N the right-hand side of (3.30) is asymptotically normal with variance $[\lambda(\psi)/\gamma(\psi,f)]^2 c^2$ where

$$c^2 = \sum_{k=1}^{s} (\underset{\sim}{t}_k - \underset{\sim}{u}AM_k^*)M_k^{*-1}(\underset{\sim}{t}_k' - M_k^*A\underset{\sim}{u}') \,, \tag{3.31}$$

and mean

$$\begin{aligned} \sum_{k=1}^{s} \underset{\sim}{b}_k^*(\underset{\sim}{t}_k' - M_k^*A\underset{\sim}{u}') &= \sum_{k=1}^{s} \underset{\sim}{b}_k^*\underset{\sim}{t}_k' - (\sum_{k=1}^{s} \underset{\sim}{b}_k^*M_k^*)A\underset{\sim}{u}' \\ &= \underset{\sim}{b}^*\underset{\sim}{t}' \,, \end{aligned} \tag{3.32}$$

the last equality in (3.32) being a consequence of (3.2). Expanding the right-hand side of (3.31) and using (3.17), we have

$$\begin{aligned} c^2 &= \sum_{k=1}^{s} \underset{\sim}{t}_k M_k^{*-1} \underset{\sim}{t}_k' - \sum_{k=1}^{s} \underset{\sim}{t}_k A\underset{\sim}{u}' \\ &= \sum_{k=1}^{s} \underset{\sim}{t}_k \sum_{j=1}^{s} (\delta_{kj}M_j^{*-1} - A)\underset{\sim}{t}_j' \\ &= \underset{\sim}{t}J\underset{\sim}{t}' \,. \end{aligned}$$

Thus, for any $\underset{\sim}{t} \in \mathbb{R}^{sq}$, $\underset{\sim}{T}_N\underset{\sim}{t}'$ under H_N has asymptotically a (possibly degenerate) normal distribution with mean $\underset{\sim}{b}^*\underset{\sim}{t}'$ and variance $[\lambda(\psi)/\gamma(\psi,f)]^2\underset{\sim}{t}J\underset{\sim}{t}'$. It follows that

$$\mathcal{D}(\underset{\sim}{T}_N|H_N) \to \mathcal{N}_{sq}(\underset{\sim}{b}^*, [\lambda(\psi)/\gamma(\psi,f)]^2 J) \,. \tag{3.33}$$

Lemma 3.6. Under H_N, the random variable

(3.34) $$Q_N^* = N[\gamma(\psi,f)/\lambda(\psi)]^2 \sum_{k=1}^{s} (\hat{\beta}_{kn_k} - \hat{\beta}_N)M_k^*(\hat{\beta}_{kn_k} - \hat{\beta}_N)'$$

is asymptotically $\chi^2_{(s-1)q}(\Delta_Q)$.

Proof. Let

(3.35) $$Y_N = [\gamma(\psi,f)/\lambda(\psi)]T_N = (Y_{N1},\ldots,Y_{Ns})$$

where

(3.36) $$Y_{Nk} = [\gamma(\psi,f)/\lambda(\psi)]N^{\frac{1}{2}}(\hat{\beta}_{kn_k} - \hat{\beta}_N)\ ,\quad k = 1,\ldots,s\ .$$

Then by (3.33) we have

(3.37) $$\mathcal{D}(Y_N|H_N) \to N_{sq}([\gamma(\psi,f)/\lambda(\psi)]b^*,J)\ .$$

Define the $(sq) \times (sq)$ symmetric matrix

(3.38) $$K = (K_{kj})_{k,j=1,\ldots,s} = (\delta_{kj}M_k^*)_{k,j=1,\ldots,s}\ .$$

Then (3.34) and (3.3) can be rewritten respectively as

(3.39) $$Q_N^* = \sum_{k=1}^{s} Y_{Nk}M_k^*Y_{Nk}' = Y_NKY_N'$$

and

(3.40) $$\Delta_Q = \{[\gamma(\psi,f)/\lambda(\psi)]b^*\}K\{[\gamma(\psi,f)/\lambda(\psi)]b^*\}'\ .$$

So, to prove Lemma 3.6, it suffices to show that KJ, or equivalently its transpose

(3.41) $$W = JK\ ,$$

is idempotent with trace equal to $(s-1)q$ and that

$$b^{*}KJKb^{*\prime} = b^{*}Kb^{*\prime} \tag{3.42}$$

(see Searle (1971), Corollary 2s.1). By direct computation we have

$$W = (W_{kj})_{k,j=1,\ldots,s} \tag{3.43}$$

where

$$W_{kj} = \delta_{kj} I_q - AM_j^{*} \quad , \quad k,j=1,\ldots,s \quad . \tag{3.44}$$

By further computation and (3.17) we have $W^2 = W$. On the other hand

$$KJK = KW = (\delta_{kj} M_k^{*} - M_k^{*} A M_j^{*})_{k,j=1,\ldots,s}$$

and so

$$\begin{aligned} b^{*}KJKb^{*\prime} &= \sum_{k=1}^{s} b_k^{*} M_k^{*} b_k^{*\prime} - \left(\sum_{k=1}^{s} b_k^{*} M_k^{*}\right) A \sum_{j=1}^{s} M_j^{*} b_j^{*\prime} \\ &= b^{*}Kb^{*\prime} , \end{aligned} \tag{3.45}$$

where the last equality in (3.45) follows from (3.2).

It remains to compute the trace of W . Let

$$M_k^{*} = (c_{km\ell})_{m,\ell=1,\ldots,q} \quad , \quad k = 1,\ldots,s \quad . \tag{3.46}$$

Then by (3.15) we have

$$d_{m\ell} = \sum_{k=1}^{s} c_{km\ell} \quad , \quad m,\ell = 1,\ldots,q \tag{3.47}$$

and by (3.16) and (3.17) we have

$$(3.48) \qquad \sum_{m=1}^{q} a_{\ell m} c_{km\ell} = 1 \quad , \qquad \ell = 1,\dots,q \quad .$$

Now for $k = 1,\dots,s$, by (3.44) we have

$$W_{kk} = I_q - AM_k^*$$

$$= (\delta_{\ell\ell'} - \sum_{m=1}^{q} a_{\ell m} c_{km\ell'}) \; , \; \ell,\ell' = 1,\dots,q \quad .$$

with trace

$$\mathrm{tr}\, W_{kk} = q - \sum_{\ell=1}^{q} \sum_{m=1}^{q} a_{\ell m} c_{km\ell} \quad .$$

It follows that

$$(3.49) \qquad \mathrm{tr}(W) = \sum_{k=1}^{s} \mathrm{tr}(W_{kk}) = sq - q = (s-1)q \quad .$$

Thus Lemma 3.6 is established.

<u>Proof of Theorem 3.1</u>. By Lemma 3.6 it suffices to show that

$$(3.50) \qquad \hat{C}_N \to C_N^* \quad .$$

By (1.12), (1.15) - (1.17) and (2.9) we have

$$\lim_{N\to\infty} n_k^{-1} \sum_{i=1}^{n_k} (a_{n_k}(i) - \bar{a}_{n_k})^2 = \lambda^2(\psi) \quad , \qquad k = 1,\dots,s$$

and so by (2.8), (1.10) and (1.13) we have

$$(3.51) \qquad \lim_{N\to\infty} \lambda_N = \lambda(\psi) \quad .$$

It follows from (3.11) and (3.36) that

$$(3.52) \qquad (N\lambda_N^2)^{-\frac{1}{2}}\hat{S}_{Nk} \to Y_{Nk}M_k^* \quad , \; k = 1,\dots,s \quad .$$

Now by (1.14) we have

$$(3.53) \qquad \lim_{N\to\infty} NM^{-1}_{kn_k} = M^{*-1}_k \quad , \quad k = 1,\dots,s \quad .$$

It follows from the symmetry of M^*_k that

$$(3.54) \qquad \lambda_N^{-2}\hat{S}_{Nk} M^{-1}_{kn_k} \hat{S}'_{Nk} \sim Y_{Nk} M^*_k Y'_{Nk} \quad .$$

Summing up both sides of (3.54) over $k = 1,\dots,s$ and using (2.10) and (3.39), we obtain (3.50). Thus Theorem 3.1 is proved.

4. Asymptotic Efficiency. Using (3.7), we rewrite (0.1) as

$$(4.1) \qquad X_{kn_k} = \alpha_k 1_{n_k} + \beta_k C_k + Z_{kn_k} \quad , \quad k = 1,\dots,s$$

where

$$(4.2) \qquad 1_{n_k} = (1,\dots,1) \in \mathbb{R}^{n_k} \; ,$$

$$(4.3) \qquad Z_{kn_k} = (Z_{k1},\dots,Z_{kn_k}) \quad ,$$

and

$$(4.4) \qquad C_k = (c_{k1},\dots,c_{kn_k})$$

is a $q \times n_k$ matrix.

Let

$$(4.5) \qquad X_N = (X_{1n_1},\dots,X_{sn_s}) \quad ,$$

$$(4.6) \qquad Z_N = (Z_{1n_1},\dots,Z_{sn_s}) \quad ,$$

(4.7) $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_s)$,

(4.8) $\underset{\sim}{\theta} = (\underset{\sim}{\alpha}, \underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_s)$,

and let

(4.9) $E_k = (\underset{\sim}{0}', \ldots, \underset{\sim}{1}'_{n_k}, \ldots, \underset{\sim}{0}')'$, $k = 1, \ldots, s$

be the $s \times n_k$ matrix with $\underset{\sim}{1}_{n_k}$ as the k-th row and all the other rows being $\underset{\sim}{0}$. Then the s linear models in (4.1) can be combined into one linear model

(4.10) $\underset{\sim}{X}_N = \underset{\sim}{\theta} C_N^* + \underset{\sim}{Z}_N$

where

(4.11)
$$C_N^* = \begin{bmatrix} E_1 & E_2 & \cdots & E_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_s \end{bmatrix}$$

is an $[s(q + 1)] \times N$ matrix. The parameter space for $\underset{\sim}{\theta}$ is the $s(q + 1)$-dimensional Euclidean space

(4.12) $$\Omega = \mathbb{R}^{s(q+1)}$$

and H_o can be expressed as

(4.13) $H_o : \underset{\sim}{\theta} \in \Omega_o = \{(\underset{\sim}{a}, \underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_s) \in \Omega : \underset{\sim}{b}_1 = \ldots = \underset{\sim}{b}_s\}$.

The likelihood ratio test of (0.5) rejects H_o (in favor of H)

if the likelihood ratio

$$\Lambda_N = \sup\{\prod_{k=1}^{s}\prod_{i=1}^{n_k} f(X_{ki} - a_k - \underset{\sim}{b}\underset{\sim}{c}_{ki}) : a_k \in \mathbb{R} , k = 1,\ldots,s; \underset{\sim}{b} \in \mathbb{R}^q\} /$$

$$\sup\{\prod_{k=1}^{s}\prod_{i=1}^{n_k} f(X_{ki} - a_k - \underset{\sim}{b}_k\underset{\sim}{c}_{ki}) : a_k \in \mathbb{R} , \underset{\sim}{b}_k \in \mathbb{R}^q , k = 1,\ldots,s\}$$

is small, or equivalently if

(4.14) $$L_N = -2 \log \Lambda_N$$

is large. Here f (or equivalently F) is assumed to be known. Under Assumptions I - V and VII of Wald (1943), but no assumption concerning the shape of F, the asymptotic distribution of L_N under H_N is given by Theorem 4.1, which will be proved later in this section.

<u>Theorem 4.1</u>. <u>Under</u> H_N, L_N <u>is asymptotically</u> $\chi^2_{(s-1)q}(\Delta_L)$ <u>with</u>

(4.15) $$\Delta_L = I(f) \sum_{k=1}^{s} \underset{\sim}{b}_k^* M_k \underset{\sim}{b}_k^{*\prime} .$$

To compare the proposed rank-order tests with the likelihood ratio test, we make the additional assumption that

(4.16) $$\underset{\sim}{b}_k^* \neq 0 \quad \text{for some} \quad 1 \leq k \leq s ,$$

which makes the right-hand side of (4.15) strictly positive. Combining Theorems 3.1 and 4.1, we have the asymptotic relative efficiency.

Corollary 4.2. The asymptotic efficiency of the aligned rank-order test of (0.5) (based on Q_N) relative to the likelihood ratio test (based on L_N) is

$$
\begin{aligned}
e_{Q,L}(F) &= \gamma^2(\psi,f) / [I(f)\lambda^2(\psi)] \\
\text{(4.17)} \qquad &= \left[\int_0^1 \psi(u)\phi_f(u)\,du\right]^2 \Big/ \left[\int_0^1 [\phi_f(u)]^2 du \int_0^1 [\psi(u) - \bar{\psi}]^2 du\right] .
\end{aligned}
$$

Clearly if the score-generating function ψ is the same as ϕ_f, then the right-hand side of (4.17) reduces to unity.

Corollary 4.3. With the score-generating function $\psi = \phi_f$, the aligned rank-order test of (0.5) has asymptotic relative efficiency one with respect to the likelihood ratio test.

Examples. If F is the standard logistic distribution function, then $\psi(u) = \phi_f(u) = 2u - 1$ generates Wilcoxon-type scores; and if $F = \Phi$ is the standard normal distribution function, then $\psi = \phi_{\Phi'} = \Phi^{-1}$ generates normal scores.

Proof of Theorem 4.1. Consider the map

$$\text{(4.18)} \qquad \underset{\sim}{\xi} = (\underset{\sim}{\xi}_0, \underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_s) : \Omega \to \Omega$$

defined by

$$
\text{(4.19)} \qquad
\begin{aligned}
&\underset{\sim}{\xi}_0(\underset{\sim}{\theta}) = \underset{\sim}{\alpha} , \quad \underset{\sim}{\xi}_1(\underset{\sim}{\theta}) = \underset{\sim}{\beta}_1 , \\
&\underset{\sim}{\xi}_k(\underset{\sim}{\theta}) = \underset{\sim}{\beta}_k - \underset{\sim}{\beta}_1 \quad \text{for} \quad k = 2,\ldots,s \quad , \; (\underset{\sim}{\theta} = (\underset{\sim}{\alpha}, \underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_s) \in \Omega) .
\end{aligned}
$$

Let

$$(4.20)\qquad \begin{aligned} \underset{\sim}{\xi}_{(1)} &= (\underset{\sim}{\xi}_0, \underset{\sim}{\xi}_1) = (\xi_1, \ldots, \xi_{s+q}) \\ \underset{\sim}{\xi}_{(2)} &= (\underset{\sim}{\xi}_2, \ldots, \underset{\sim}{\xi}_s) = (\xi_{s+q+1}, \ldots, \xi_{s(q+1)}) \ . \end{aligned}$$

Then H_0 can be expressed as

$$(4.21)\qquad \underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta}) \equiv \underset{\sim}{0} \ .$$

Clearly $\underset{\sim}{\xi}$ is a homeomorphism and, with the identification

$$(4.22)\qquad \underset{\sim}{\theta} = (\underset{\sim}{\alpha}, \underset{\sim}{\beta}_1, \ldots, \underset{\sim}{\beta}_s) = (\theta_1, \ldots, \theta_{s(q+1)}) \ ,$$

has a positive Jacobian $\det(\partial\underset{\sim}{\xi}/\partial\underset{\sim}{\theta})$ not depending on $\underset{\sim}{\theta}$; moreover, the first two partial derivatives of $\xi_1(\underset{\sim}{\theta}), \ldots, \xi_{s(q+1)}(\underset{\sim}{\theta})$ are uniformly continuous and bounded functions of $\underset{\sim}{\theta}$. Indeed, the inverse $\underset{\sim}{\theta} = \underset{\sim}{\xi}^{-1}$ of $\underset{\sim}{\xi}$ is given by

$$(4.23)\qquad \underset{\sim}{\theta}(\underset{\sim}{\xi}) = (\underset{\sim}{\xi}_0, \underset{\sim}{\xi}_1, \underset{\sim}{\xi}_1 + \underset{\sim}{\xi}_2, \ldots, \underset{\sim}{\xi}_1 + \underset{\sim}{\xi}_s)$$

with Jacobian matrix

$$(4.24)\qquad M = (\partial\underset{\sim}{\theta}/\partial\underset{\sim}{\xi}) = \begin{bmatrix} I_s & 0 & 0 \ldots & 0 \\ 0 & I_q & 0 \ldots & 0 \\ 0 & I_q & I_q \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & I_q & 0 \ldots & I_q \end{bmatrix}$$

(whose determinant is equal to unity).

Now consider the $[s(q+1)] \times [s(q+1)]$ matrix

$$(4.25)\qquad A_N = C_N^* C_N^{*\prime} = \begin{bmatrix} E & F_1' & F_2' & \cdots & F_s' \\ F_1 & D_1 & 0 & \cdots & 0 \\ F_2 & 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ F_s & 0 & 0 & \cdots & D_s \end{bmatrix}$$

where

$$(4.26)\qquad E = (\delta_{kj} n_k)_{k,j=1,\ldots,s}$$

$$(4.27)\qquad D_k = C_k C_k' = \sum_{i=1}^{n_k} \underset{\sim}{c}_{ki} \underset{\sim}{c}_{ki}' \quad , \quad k = 1,\ldots,s$$

and

$$(4.28)\qquad F_k = (\underset{\sim}{0}',\ldots,n_k \underset{\sim}{\bar{c}}_{kn_k},\ldots,\underset{\sim}{0}') \ , \quad k = 1,\ldots,s$$

is a $q \times s$ matrix with $n_k \underset{\sim}{\bar{c}}_{kn_k}$ as the k-th column and 0 elsewhere. By routine computation we have the $[s(q + 1)] \times [s(q + 1)]$ matrix

$$(4.29)\qquad A_N^* = M' A_N M = \begin{bmatrix} A_{N11}^* & A_{N12}^* \\ A_{N21}^* & A_{N22}^* \end{bmatrix}$$

where

$$(4.30)\qquad A_{N11}^* = \begin{bmatrix} E & \sum_{k=1}^{s} F_k' \\ \sum_{k=1}^{s} F_k & \sum_{k=1}^{s} D_k \end{bmatrix}$$

is $(s + q) \times (s + q)$,

$$A^*_{N12} = \begin{bmatrix} F'_2 & \cdots & F'_s \\ D_2 & \cdots & D_s \end{bmatrix} \tag{4.31}$$

$$A^*_{N21} = A^{*\prime}_{N12} \quad , \tag{4.32}$$

and

$$A^*_{N22} = (\delta_{kj} D_k)_{k,j=2,\ldots,s} \quad . \tag{4.33}$$

We note that by assumption A_N is positive definite, hence so is A^*_N. Consider the matrix

$$M_N = \sum_{k=1}^{s} M_{kn_k} \quad , \tag{4.34}$$

which is symmetric and positive definite and hence has a symmetric inverse

$$G_N = M_N^{-1} \quad , \tag{4.35}$$

and define the $q \times s$ matrix

$$\bar{C}_N = (\bar{c}_{1n_1}, \ldots, \bar{c}_{sn_s}) \quad . \tag{4.36}$$

Then by routine computation and the obvious identity

$$M_{kn_k} = D_k - n_k \bar{c}_{kn_k} \bar{c}'_{kn_k} \quad , \quad k = 1, \ldots, s \tag{4.37}$$

it can be checked that

$$A^{*-1}_{N11} = \begin{bmatrix} \bar{C}'_N G_N \bar{C}_N + E^{-1} & -\bar{C}'_N G_N \\ -G_N \bar{C}_N & G_N \end{bmatrix} \tag{4.38}$$

By further computation and the additional identity

$$(4.39) \qquad F_k E^{-1} F_j' = \delta_{kj} n_k \bar{\underset{\sim}{c}}_{kn_k} \bar{\underset{\sim}{c}}'_{jn_j} \quad , \qquad k,j = 1,\ldots,s$$

we have

$$(4.40) \qquad A^*_{N21} A^{*-1}_{N11} A^*_{N12} = (\delta_{kj} n_k \bar{\underset{\sim}{c}}_{kn_k} \bar{\underset{\sim}{c}}'_{jn_j} + M_{kn_k} G_N M_{jn_j})_{k,j=2,\ldots,s} \; .$$

So we have the $[(s - 1)q] \times [(s - 1)q]$ matrix

$$(4.41) \qquad \bar{A}^*_N = A^*_{N22} - A^*_{N21} A^{*-1}_{N11} A^*_{N12}$$
$$= (\delta_{kj} M_{kn_k} - M_{kn_k} G_N M_{jn_j})_{k,j=2,\ldots,s} \; .$$

Now H_N can be expressed as

$$(4.42) \qquad H_N : \underset{\sim}{\theta} = \underset{\sim}{\theta}_N = (\underset{\sim}{\alpha}, \underset{\sim}{\beta}_{1N}, \ldots, \underset{\sim}{\beta}_{sN}) \; .$$

We also note that

$$(4.43) \qquad \underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta}_N) = N^{-\frac{1}{2}}[(\underset{\sim}{b}^*_2, \ldots, \underset{\sim}{b}^*_s) - (\underset{\sim}{b}^*_1, \ldots, \underset{\sim}{b}^*_1)] \; .$$

Then, by Theorem IX of Wald (1943), L_N under H_N is asymptotically noncentral chi-square with $s(q + 1) - (s + q) = (s - 1)q$ degrees of freedom and noncentrality parameter

$$(4.44) \qquad \Delta_{L_N} = I(f) \underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta}_N) \bar{A}^*_N \underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta}_N)' = I(f) \Delta^*_N$$

where

$$(4.45) \qquad \Delta^*_N = N^{-1}[(I) - (II) - (III) + (IV)]$$

with

$$\text{(4.46)}\quad \begin{aligned}(\mathrm{I}) &= (\underset{\sim}{b}_2^*,\ldots,\underset{\sim}{b}_s^*)\overline{A}_N^*(\underset{\sim}{b}_2^*,\ldots,\underset{\sim}{b}_s^*)' \\ &= \sum_{k=2}^{s} \underset{\sim}{b}_k^* M_{kn_k}\underset{\sim}{b}_k^{*\prime} - (\sum_{k=2}^{s} \underset{\sim}{b}_k^* M_{kn_k}) G_N (\sum_{k=2}^{s} M_{kn_k}\underset{\sim}{b}_k^{*\prime}) ,\end{aligned}$$

$$\text{(4.47)}\quad \begin{aligned}(\mathrm{II}) &= (\underset{\sim}{b}_2^*,\ldots,\underset{\sim}{b}_s^*)\overline{A}_N^*(\underset{\sim}{b}_1^*,\ldots,\underset{\sim}{b}_1^*)' \\ &= (\sum_{k=1}^{s} \underset{\sim}{b}_k^* M_{kn_k}) G_N M_{1n_1}\underset{\sim}{b}_1^{*\prime} ,\end{aligned}$$

$$\text{(4.48)}\qquad (\mathrm{III}) = (\mathrm{II})'$$

and

$$\text{(4.49)}\quad \begin{aligned}(\mathrm{IV}) &= (\underset{\sim}{b}_1^*,\ldots,\underset{\sim}{b}_1^*)\overline{A}_N^*(\underset{\sim}{b}_1^*,\ldots,\underset{\sim}{b}_1^*)' \\ &= \underset{\sim}{b}_1^* M_{1n_1}\underset{\sim}{b}_1^{*\prime} - \underset{\sim}{b}_1^* M_{1n_1} G_N M_{1n_1}\underset{\sim}{b}_1^{*\prime} .\end{aligned}$$

By (1.14) and (3.2) we have

$$\text{(4.50)}\qquad \lim_{N\to\infty} N^{-1} \sum_{k=2}^{s} \underset{\sim}{b}_k^* M_{kn_k} = -\underset{\sim}{b}_1^* M_1^* .$$

And by (3.15) - (3.16) and (4.34) - (4.35) we have

$$\text{(4.51)}\qquad \lim_{N\to\infty} NG_N = A .$$

It follows that

$$\lim_{N\to\infty} \Lambda_N^* = \sum_{k=1}^{s} \underset{\sim}{b}_k^* M_k^* \underset{\sim}{b}_k^{*\prime} \text{ and so } \lim_{N\to\infty} \Delta_{L_N} = \Delta_L .$$

The proof is complete.

5. Asymptotic Optimality. Let Γ_{N1} and B_N be non-singular square matrices of orders $s + q$ and $(s - 1)q$, respectively, satisfying

(5.1) $$\Gamma_{N1}'\Gamma_{N1} = A^*_{N11}$$

and

(5.2) $$B_N'B_N = \bar{A}^*_N \quad ,$$

and define the $(s + q) \times [(s - 1)q]$ matrix

(5.3) $$\Gamma_{N2} = (\Gamma_{N1}')^{-1}A^*_{N12} \quad .$$

Then the square matrix

(5.4) $$K_N = \begin{bmatrix} \Gamma_{N1} & \Gamma_{N2} \\ 0 & B_N \end{bmatrix}$$

of order $s(q + 1)$ is nonsingular and satisfies

(5.5) $$K_N A_N^{*-1} K_N' = I_{s(q + 1)} \quad .$$

For $\underset{\sim}{\omega} = (\underset{\sim}{a}, \underset{\sim}{b}, \ldots, \underset{\sim}{b}) \in \Omega_0$ and $c > 0$ define the surface

(5.6) $S(\underset{\sim}{\omega}, c)$
$$= \{\underset{\sim}{\theta} \in \Omega : I(f)\underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta})\bar{A}^*_N\underset{\sim}{\xi}_{(2)}(\underset{\sim}{\theta})' = c, \ \underset{\sim}{\xi}(\underset{\sim}{\theta})\Gamma_N' = (\underset{\sim}{a}, \underset{\sim}{b})\Gamma_{N1}'\}$$

where

(5.7) $$\Gamma_N = (\Gamma_{N1}, \Gamma_{N2})$$

is $(s + q) \times [s(q + 1)]$. Consider the transformation of Ω

(5.8) $\quad \theta = (\alpha, \beta_1, \ldots, \beta_s) \rightarrow (\alpha^*, \beta_1^*, \ldots, \beta_s^*) = [I(f)]^{\frac{1}{2}} \xi(\theta) K_N'$

where

(5.9)
$$(\alpha^*, \beta_1^*) = [I(f)]^{\frac{1}{2}} \xi(\theta) \Gamma_N'$$

and

(5.10)
$$(\beta_2^*, \ldots, \beta_s^*) = [I(f)]^{\frac{1}{2}} \xi_{(2)}(\theta) B_N' ,$$

which maps $S(\omega, c)$ into

(5.11)

$$S^*(\omega, c)$$
$$= \{(\alpha^*, \beta_1^*, \ldots, \beta_s^*) \in \Omega : (\alpha^*, \beta_1^*) = [I(f)]^{\frac{1}{2}}(a, b) \Gamma_{N1}', \sum_{k=2}^{s} \beta_k^* \beta_k^{*\prime} = c\} .$$

For $\theta_0 \in \Omega$ and $\rho > 0$ let

(5.12)

$$\Omega(\theta_0, \rho)$$
$$= \{\theta \in \Omega : \theta, \theta_0 \in S(\omega, c) \text{ for some } \omega \in \Omega_0 \text{ and } c > 0, \text{ and } \|\theta - \theta_0\| \leq \rho\}$$

($\| \quad \|$ being the Euclidean norm on Ω) , and let $\Omega^*(\theta_0, \rho)$ be its image under the transformation (5.8). For $\theta \in \Omega$ let

(5.13)
$$\eta(\theta) = \lim_{\rho \to 0} \{A[\Omega^*(\theta, \rho)] / A[\Omega(\theta, \rho)]\}$$

where A denotes area. Then by Theorem VIII of Wald (1943) the likelihood ratio test of (0.5) is asymptotically optimal in the sense that it

(a) has asymptotically best average power with respect to the weight function $\eta(\underset{\sim}{\theta})$ and the family of surfaces

(5.14) $$\mathcal{S} = \{S(\underset{\sim}{\omega}, c) : \underset{\sim}{\omega} \in \Omega_0 , c > 0\} ;$$

(b) has asymptotically best constant power on the surfaces in $\mathcal{S}$;

and

(c) is an asymptotically most stringent test.

By Corollary 4.3., with the score-generating function $\psi = \phi_f$ the proposed rank-order test is asymptotically power-equivalent to the Wald-optimal likelihood ratio test. Thus if the underlying distribution F is logistic, then the Q_N-test using Wilcoxon-type scores is asymptotically optimal; and if F is normal, then the normal-scores rank-order test is asymptotically optimal.

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic distribution, asymptotic efficiency, asymptotic optimality, linear regression, parallelism, rank-order tests, regression surfaces.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For testing the hypothesis that several $(s \geq 2)$ linear regression surfaces $X_{ki} = \alpha_k + \underset{\sim}{\beta}_k \underset{\sim}{c}_{ki} + Z_{ki}$ $(k = 1,\dots,s)$ are parallel to one another, i.e. $\underset{\sim}{\beta}_1 = \dots = \underset{\sim}{\beta}_s$, a class of rank-order tests are considered. The tests are shown to be asymptotically distribution-free, and their asymptotic efficiency relative to the general likelihood ratio test is derived. Asymptotic optimality in the sense of Wald is also discussed.

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